Transport and localization of waves in one-dimensional disordered media : Random phase approximation and beyond

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We report a systematic and detailed numerical study of statistics of the reflection coefficient ($|R(L)|^2$) and its associated phase (θ) for a plane wave reflected from a one-dimensional (1D) disordered medium beyond the random phase approximation (RPA) for Gaussian white-noise disorder. We solve numerically the full Fokker-Planck (FP) equation for the probability distribution in the ($|R(L)|^2, \theta(L)$)-space for different lengths of the sample with different "disorder strengths". The statistical electronic transport properties of 1D disordered conductors are calculated using the Landauer four-probe resistance formula and the FP equation. This constitutes a complete solution for the reflection statistics and many aspects of electron transport in a 1D Gaussian white-noise potential. Our calculation shows the contribution of the phase distribution to the different averages and its effects on the one-parameter scaling theory of localization.

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I. INTRODUCTION

Wave propagation through 1D random media have been studied for decades. Typical physical examples of these problems are electron transport in disordered conductors, light transport in random dielectric media, etc.. All previous studies agree that the transmission through a 1D disordered medium decays exponentially with the length of the sample. For electronic systems, the conductance decays exponentially with the length of the sample and the resistance increases exponentially with the sample length. The main characteristic length scale in the problem is the localization length. Results of different phases of research in this field have been reviewed recently in several review articles [1–7].

Disordered 1D systems are quite generic as regards their transport properties — disordered 1D conductors are all alike, but every ordered 1D conductor is ordered in its own way! Thus, Mott *et. al.* [1] first showed that all the eigenstates of 1D disordered quantum systems are exponentially localized. There are no truly 1D disordered metals. For electronic systems quantum interference effects are important at low temperatures when decoherence due to the inelastic process may be neglected. At low enough temperatures with static disorder, the transport properties of the electronic systems are highly fluctuating (i.e. sample specific) due to the different conductors with the same impurity concentration but with different microscopic arrangements of the impurities, can differ substantially in their transport properties. Several studies show that the root mean-squared-fluctuation is more than the average for length scales larger than the localization length, while for the length scales within the localization length (i.e. in the good metallic regime for quasi 1D and higher dimensional systems) the fluctuations are finite and universal, the so called universal conductance fluctuation (UCF) [7]. These fluctuations make the resistance and the conductance non-self-averaging quantities. The resistance at low temperatures when quantum effects are important is non-Ohmic, i.e., non additive in series. The non-additive nature arises because of the non-local effects of the quantum wave amplitudes associated with the electron in the conductor. One has to take proper care of the phases to add the two quantum resistances together.

To calculate any meaningful quantity for a non-self-averaging quantity, like the resistance and the conductance, one should know the full probability distribution of the same for every length of the sample. Several authors [8–12] have derived a Fokker-Planck equation for the full probability distribution of the resistance for the Gaussian white noise disorder. Exact results for the average resistance and the average conductance can be calculated analytically for the weak-disorder case using the Fokker-Planck equation. The approximation made in the weak-disorder case is that the phase of the complex reflection amplitude relative to the incident wave, or the relative phase of two non-ohmically additive quantum resistances, are distributed uniformly, which is the random phase approximation (RPA). Then the problem can be solved analytically to calculate the average resistance exactly for the Gaussian white noise potential. The analytical results show that the resistance and the conductance have log-normal distributions for large lengths. The average four-probe resistance increases exponentially with the length of the sample with a characteristic length scale called the localization length. The average four-probe conductance, however, diverges for all length scales. The divergence of the average conductance for all length scales is due to resonance (absolute transmission) states which dominate the probability distribution [13,14]. We will discuss elsewhere the issues related to the divergence of the average conductance.

It is difficult to get an average quantity by direct ana-

lytical calculation which includes the phase distribution, where the phase distribution depends on the strength of the disorder. However, the actual validity of the random phase approximation has not been studied in detail. There are issues like actual contribution of the phase distribution to the different averaging processes that have not been systematically studied so far.

We study numerically, the joint probability distribution of the phase (θ) and the reflection coefficient (r) of the complex amplitude reflection coefficient ($R(L) = \sqrt{r}\, exp(i\theta)$) for a one-dimensional disordered conductor with the Gaussian white noise disorder, for different lengths of the sample and different strengths of the disorder. This is mainly the solution of the Fokker-Planck equation, where the probability density is varying in the (r, θ)-space and evolving with the sample length. Using the "invariant imbedding" technique, a non-linear Langevin equation can be derived for the $R(L)$. Then, using the stochastic Liouville equation for the probability evolution and the Novikov theorem to integrate out the stochastic aspect due to the random potential, one gets the Fokker-Planck (FP) equation in the (r, θ)-space for varying sample length L [10,12]. Integrating θ and r parts separately from the joint probability distribution $P(r, \theta)$, we have calculated the marginal probability distributions for r and θ, respectively, for a given length and disorder strength of the sample. The Landauer formula [15,16] relates the resistance and the conductance directly to the reflection coefficient (r). Once the reflection probability is known, the average electronic properties of the system such as resistance, conductance *etc.* can be calculated easily using the Landauer formula. We have outlined the range of validity of the random phase approximation, for the parameters on which the phase distribution depends, namely the localization length and the wave vector of the incoming wave. Our calculation shows that $\xi \geq \lambda = 2\pi/k$ is the limit where the random phase approximation is valid, where ξ is the localization length and λ is the wave length of the incoming wave (and for the electronic case we will consider λ to be Fermi wave length)

Knowing the probability distribution of the reflection coefficient and using the Landauer four-probe resistance/conductance formula, we have calculated the probability distribution for the resistance/conductance and find it to be log-normal, for all strengths of the white noise Gaussian disorder. We calculate the averages and the fluctuations of the reflection coefficient, the conductance and the resistance and of logarithms of the conductance and the resistance. Lastly, we study the effects of the phase distribution on the one-parameter scaling theory of localization. We have shown that the one-parameter scaling theory holds quite well for large sample lengths, but the phase has a definite but small effect on it. To the best of our knowledge this is the first work where the joint probability distribution for the reflection coefficient and its phase for a 1D Gaussian white-noise disordered conductor is calculated for all length scales and for different disorder strengths. The way we have calculated the different quantities here involves essentially no approximations. We will discuss works of others while discussing our results. We report here mainly the results of the electronic transport in disorder conductors, but some parts of the formalism apply equally well to the case of light transport in random dielectric media.

II. METHOD OF CALCULATION

Both the Schrödinger and the Maxwell equations can be transformed to the Helmholtz equation. Therefore, these two equations can be studied in the same frame work for the reflection and the transmission amplitude, since they transform to the same Helmholtz equation and have similar initial conditions.

Consider the Schrödinger Equation,

$$-\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = k^2\psi, \tag{1}$$

where we have set $\hbar^2/2m = 1$,
and the Maxwell equation,

$$\frac{\partial^2 E}{\partial x^2} + \frac{\omega^2}{c^2}[\epsilon_0 + \epsilon(x)]E = 0\,. \tag{2}$$

Transforming the Schrödinger and the Maxwell equation to the standard Helmholtz equation form, we obtain :

$$\frac{\partial^2 u}{\partial x^2} + k^2\left[1 + \eta(x)\right] u = 0, \tag{3}$$

where $\eta(x)$=$-V(x)/k^2$ for Schrödinger wave, and $\eta(x)$= $+\frac{\epsilon(x)}{\epsilon_0}$ and $k^2 = \frac{\omega^2}{c^2}\epsilon_0$ for the Maxwell wave, where ϵ_0 is the constant dielectric background and $\epsilon(x)$ is the spatially fluctuating part of the dielectric medium.

A. The Langevin Equation for the complex amplitude reflection coefficient R(L)

Consider a plane wave of wave vector k incident from the right side of the disordered sample of length L having the reflection amplitude $R(L)$. The non-linear Langevin equation for the reflection amplitude for the plane wave scattering problem can be derived by the invariant imbedding technique [12]. The Langevin equation for the complex amplitude reflection coefficient is :

$$\frac{dR(L)}{dL} = 2ikR(L) + i\frac{k}{2}\eta(L)(1 + R(L))^2, \tag{4}$$

with the initial condition R(L)=0 for L=0.

The main idea is to use the Langevin equation to get a FP equation for the reflection probability density. For

the electronic system consider the reflection coefficient (or the transmission coefficient) of a free electron wave scattered by the random potential in order to calculate the resistance/ conductance of the system. The average properties of the resistance/conductance can be calculated using the Landauer four-probe formula and the FP equation for the reflection coefficient.

B. The Fokker-Planck equation

To get the Fokker-Planck equation from the non-linear Langevin equation [12], one has to get the probability density equation first and then the stochastic aspect due to the random potential has to be integrated out. The Langevin equation (Eq.4) can be solved analytically for the Gaussian white noise potential to get the FP equation. The non-linear Langevin equation for $R(L)$ in (Eq.4) is basically two coupled differential equations for the magnitude and the associated phase parts.
Taking

$$R(L) = \sqrt{r(L)}exp(i\theta(L)), \tag{5}$$

and substituting Eq.5 into Eq.4, and equating the real and the imaginary parts on both sides of the Eq.4, one gets,

$$\frac{dr}{dL} = k\eta(L)r^{\frac{1}{2}}(1-r)\sin\theta, \tag{6}$$

$$\frac{d\theta}{dL} = 2k + \frac{k}{2}\eta(L)\left[2+\cos\theta(r^{1/2}+r^{-1/2})\right]. \tag{7}$$

Now, according to the van-Kampen lemma [17], these two stochastic coupled differential equations will produce a flow of the density $Q(r,\theta)$ in the (r,θ)-space according to the stochastic Liouville equation with increasing length of the sample, i.e., $Q(r,\theta)$ is the solution of the stochastic Liouville equation:

$$\frac{\partial Q(r,\theta,L)}{\partial L} = -\frac{\partial}{\partial r}(Q\frac{dr}{dL}) - \frac{\partial}{\partial\theta}(Q\frac{d\theta}{dL}), \tag{8}$$

where dr/dL and $d\theta/dL$ are given by Eqs.6 and 7. Now, substituting the values of dr/dL and $d\theta/dL$ in the Eq.8 one gets,

$$\begin{aligned}\frac{\partial Q}{\partial L} &= -k\sin\theta\frac{d}{dr}\left[r^{1/2}(1-r)\eta(L)Q\right] - k\frac{\partial}{\partial\theta}\left[\eta(L)Q\right]\\ &\quad -2k\frac{\partial Q}{\partial\theta} - \frac{k}{2}(r^{1/2}+r^{-1/2})\frac{\partial}{\partial\theta}\left[\cos\theta\eta(L)Q\right].\end{aligned} \tag{9}$$

To get the Fokker-Planck equation, Eq.9 has to be averaged over the stochastic aspect, i.e., over all realizations of the random potential. For the case of a Gaussian white noise potential, Eq.9 can be averaged out over the stochastic potential analytically using Novikov's [18] theorem. For the Gaussian white noise potential:

$$<\eta(L)>= 0 \;\; and \;\; <\eta(L)\eta(L')>= q\delta(L-L'). \tag{10}$$

Eq.(9) has terms like ηQ which are to be averaged out. For the Gaussian white noise disorder, the Novikov theorem says :

$$<\eta(L)Q[\eta]>_\eta = \frac{q}{2} < \frac{\delta Q[\eta]}{\delta\eta(L)} > . \tag{11}$$

After averaging out the disorder aspect in Eq.9 and writing $< Q(r,\theta) >_\eta \equiv P(r,\theta)$, one gets the Fokker-Planck equation:

$$\begin{aligned}\frac{\partial P(r,\theta)}{\partial l} &= \left[\sin\theta\frac{\partial}{\partial r}r^{1/2}(1-r) + \frac{\partial}{\partial\theta}\right.\\ &\quad \left. + \frac{1}{2}(r^{1/2}+r^{-1/2})\frac{\partial}{\partial\theta}\cos\theta\right]^2 P(r,\theta)\\ &\quad - 2k\xi\frac{\partial P(r,\theta)}{\partial\theta}.\end{aligned} \tag{12}$$

Where $l \equiv L/\xi$ and $\xi \equiv (\frac{1}{2}qk^2)^{-1}$ is the localization length.

The Fokker-Planck equation, Eq.12 has all the information of the probability distribution of the reflection coefficient (r) and the associated phase (θ) for different length scales of the sample and with varying disorder strengths. We will explore several aspects of the Eq12 in our study, which we are going to discuss in detail in later Sections.

III. PARAMETERS OF THE PROBLEM

The Fokker-Planck equation (Eq.12) has three parameters to describe the problem fully: (1) the length of the sample L, (2) the localization length ξ, and (3) the incident wave vector k. In the re-arranged form of the Eq.12, as it is written, it has effectively two parameters: $l = L/\xi$ and $C = 2k\xi$. Here l is a number which gives the length of the sample in units of the localization length and C is a number which fixes inverse of the disorder strength in terms of the wave vector of the incident wave and the localization length. Larger value of C implies that the localization length is large, or the incoming electron energy is higher, or both, that is, the weak-disordered regime. Conversely, when C is small it means ξ is small, or the incoming wave energy is small or both, that is, the strong disorder regime.

IV. ANALYTICAL SOLUTIONS FOR THE RESISTANCE AND THE CONDUCTANCE IN THE RANDOM PHASE APPROXIMATION (RPA)

A. Resistance

In the random phase approximation (RPA), which is valid for weak disorder and large incident electron energies, one can write $P(r,\theta) = (1/2\pi)P(r)$ i.e. $P(r,\theta)$ factorizes, and θ is uniformly distributed over 2π. Considering $\partial P/\partial\theta = 0$, the Fokker-Planck equation Eq.12 in r then becomes:

$$\frac{\partial P(r)}{\partial l} = \frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(1-r)^2 P(r)\right]. \tag{13}$$

(Here we have used the same symbol $P(r)$ for the marginal probability density of r as for the joint density $P(r,\theta)$.)

Now, from the Landauer four-probe resistance formula, the dimensionless resistance as a function of the reflection coefficient is:

$$\rho(l) = \frac{r(l)}{1-r(l)}. \tag{14}$$

Making the transformation from $P(r)$ to $P(\rho)$ by using $P(\rho) = P(r)(dr/d\rho)$, one obtains the Fokker-Planck equation for the resistance:

$$\frac{\partial P(\rho)}{\partial l} = \frac{\partial}{\partial\rho}\rho(1+\rho)\frac{\partial}{\partial\rho}P(\rho), \tag{15}$$

with the initial condition, $P(\rho) = \delta(\rho)$ for $l = 0$.

Eq.15 has been derived earlier by several authors [19,10,12]. This equation has also been derived from the maximum entropy principle (MEP) by Mello and Kumar [20].

The average of ρ^n can be obtained without solving directly Eq.15 as following.
Let us define,

$$\rho_n = \int_0^\infty P(\rho)\rho^n d\rho \tag{16}$$

Multiplying Eq.15 by ρ^n and integrating both sides of the equation for ρ from 0 to ∞, one gets a moment recursion equation for the average moments of the resistance,

$$\frac{d\rho_n}{dl} = n(n+1)\rho_n + n^2\rho_{n-1}. \tag{17}$$

Since the probability is always normalizable, the value of ρ_0 will be:

$$\rho_0 = \int_0^\infty P(\rho)\rho^0 = \int_0^\infty P(\rho) = 1 \tag{18}$$

Once we know the initial value ρ_0, then Eq.17 can be solved analytically for higher moments of the average resistance,

$$\rho_1(l) = \frac{1}{2}(e^{2l} - 1), \tag{19}$$

$$\rho_2(l) = \frac{1}{12}(2e^{6l} - 6e^{2l} + 4), \tag{20}$$

$$\rho_n(l) \simeq \frac{e^{n(n+1)l}}{n(n+1)!}, \;\; n \gg 1 \tag{21}$$

and specifically,

$$\text{rms fluctuation of } \rho \sim e^{3l}. \tag{22}$$

The above expressions imply that the average resistance increases exponentially with the length of the sample and the rms fluctuation is greater than the average, indicating that the resistance is not a self-averaging quantity. It is clear now why one has to consider the full probability distribution to describe the disordered quantum resistors. Eq.15 has also an analytical expression for the full distribution of ρ for the large l limit [10], which is log-normal.

B. Conductance

If one solves the Fokker-Planck equation for $< g^n >$ $(=< 1/\rho^n >)$, it has been shown by several workers and explicitly by Melnikov [19] that all the moments ν of $< g^\nu >$ diverge for $\nu > 1/2$, for all length scales of the sample. The cause is the existence of the resonant states, i.e., states $r \sim 0$. Though, the probability is less for the resonant states, but it is finite. The existence of the resonance states have been discussed in detail in Ref. [13,14].

V. AIM OF OUR WORK

The main aim of our work is to evolve the full Fokker-Planck equation, (Eq.12), numerically. Eq.12 has full informations of the the statistics of the reflection coefficient and its phase for the case of Gaussian white noise disordered potential. We will study several aspects of the solution, and try to see the actual contribution of the phase, which goes beyond the random phase approximation. Important point to note here is that the numerical calculation part involves only the solution of the Fokker-Planck equation which has been derived analytically without approximation.

VI. NUMERICAL DETAILS

We took r and θ as Cartesian variables in a two-dimensional 50×50 grids. An explicit finite-difference scheme [21,22] was used to solve the FP equation. The von-Neumann stability criterion was checked and the

Courant condition for the used discrete iterative length was strictly maintained. A few results were also checked by using the rather time consuming implicit finite-difference scheme.

A. Allowed error bars

Error bars of the order of 10^{-4} for r, 3×10^{-3} for θ, and 10^{-12} for length l, were allowed for the whole range of numerical calculations.

B. Initial probability distribution $P(r, \theta)$ at $L = 0$

The FP equation (12) poses an initial value problem. The initial probability distribution $P(r, \theta)$ at $l = 0$ has to be specified, which will then evolve with the increase of the length of the sample. The Fokker-Planck equation (12) is however singular at r=0. This causes a technical problem for solving the equation numerically. To circumvent this problem, we have therefore taken an initial (fixed) scatterer with $r = .01$ by putting a half-delta function potential peaked at $l = 0$ which could be physically understood as due to an initial impurity sitting at $l = 0$, or the contact resistance of the leads. By "fixed" we mean that it is fixed over all the realizations of the sample randomness. Phase distribution for such a weak delta-function potential will peak around $+\pi/2$ or $-\pi/2$, depending on the sign of the delta-function potential. Once $r_0 = .01$ is fixed, then the phase distribution has equal probability peak at $+\pi/2$ and $-\pi/2$. A fixed weak delta scatterer at the position $l = 0$ will not change the gross statistics, except at very smaller length scales. We have kept this initial distribution same throughout the numerical calculations. We could not consider any smaller value of the initial-fixed-reflection coefficient (r_0), or a lower cut-off to the $r = 0$ singularity, for reason of convergence criterion of the numerical algorithm. An estimate can be done for the initial cut-off length l_0 of the sample (in terms of the localization length ξ) for this small $r = r_0$. Taking analytical results for the weak disordered case, one gets:

$$\rho_0 = \frac{r_0}{(1 - r_0)} = \frac{1}{2}[exp(2l_0) - 1] \tag{23}$$

then

$$l_0 = L_0/\xi = 2ln[\frac{1 + r_0}{1 - r_0}] \simeq \frac{1}{2}ln(1 + 2r_0) \simeq r_0 = .01 \tag{24}$$

This implies that the initial length is 1% of the localization length, throughout the numerical calculation.

For numerical calculation, the delta-function has to be taken as the limit of a continuous function. In (r, θ)-space for a physically reasonable initial probability distribution, we have taken this as

$$P(r, \theta)_{l=0} = \delta(r - .01)\,[\delta(\theta - \pi/2) + \delta(\theta + \pi/2)]\,, \tag{25}$$

where the delta functions are sharp Gaussians.

Fig.1 shows this initial probability distribution $P(r, \theta)$ nominally at $l = 0$, and the marginal distribution $P(r)$ of the reflection coefficient and the marginal distribution $P(\theta)$ of the phase θ. It should be emphasized again here that the initial probability distribution of the phase θ can be taken at any small enough length. However, the statistical properties of the system do not depend on the initial distribution except for very small lengths.

C. Boundary conditions for r and θ for any length L

The unit step length of the discrete evolution is taken to be $\triangle l = 10^{-6}$. For every discrete evolution, we took $P(r, \theta) = 0$ for $r > 1$ and $r < 0$ along r axis; and the boundary condition was taken as periodic along θ axis, such that $P(r, 2\pi + \triangle\theta) = P(r, \triangle\theta)$ for every discrete evolution.

VII. RESULTS AND DISCUSSIONS

A. Evolution of $P(r, \theta)$ with the length L for different disorder strengths

We will consider the evolution of the full probability distribution $P(r, \theta)$ for different lengths l for the three main regimes of disorder strength —(1) weak, (2) medium, and (3) strong.

1. $P(r, \theta)$ for weak disorder : $2k\xi = 100$ ($2k\xi >> 1$)

Figs.2(a),(b),(c),(d) show typical distributions of the probability in (r, θ)-space for different lengths $l = 1, 2, 5, \text{and}\, 10$. Evolution shows clearly that the phase part is uniformly distributed. Evolution of the probability is mainly along the amplitude (r) axis. Probability evolves with length and peaks near $r = 1$ for large lengths. These evolution pictures imply that the random (uniform) phase distribution holds for the weak disorder, or for the large values of the localization length. Later we show the nearly exact range for the disorder parameter ($2k\xi$) where the random phase approximation is valid.

2. $P(r, \theta)$ for medium disorder: $2k\xi = 1$ ($2k\xi \sim 1$)

Similarly, Figs.3 (a),(b),(c),(d) are as Fig.2 but for the disorder parameter $2k\xi = 1$, i.e., medium disorder regime. These evolutions clearly show that the probability distribution $P(r, \theta)$ is quite complicated and has an asymmetric distribution in phase. The phase distribution peaks on one side ($\theta > \pi$) for large lengths.

3. $P(r,\theta)$ for strong disorder : $2k\xi = .001$ $(2k\xi << 1)$

Similarly, Figs.4(a),(b),(c),(d) are as Fig.2, but for the disorder parameter $2k\xi = .001$, i.e., the strong disorder regime. Probability distribution is perfectly symmetric centered at the phase $\theta = \pi$. In the strong disorder parameter regime the sample tries to behave as a perfect reflector and totally reflect back the incoming wave with opposite phase. Evolution pictures show that $P(r,\theta)$ peaks around $r = 1$ and is symmetrically around $\theta = \pi$ for larger lengths. It will be shown later that the distribution does not change substantially with further increase of the disorder strength parameter, i.e. the distribution is insensitive to the disorder parameter $2k\xi$ in that regime.

B. Marginal distribution $P(r)$ of the reflection coefficient (r) and Marginal distribution $P(\theta)$ of the phase (θ) for strong, medium and weak disorders

Figs.5(a),(b),(c) show (i) the marginal probability distribution $P(r)$ when the phase(θ) part of the $P(r,\theta)$ is integrated out and (ii) corresponding marginal distributions $P(\theta)$ of the phase θ when r part of the full distribution $P(r,\theta)$ is integrated out for the three cases of disorder regimes considered above are the (a) weak , (b) medium and (c) strong.

Fig.5 clearly indicate that the phase (θ) distributions are very different for the three different regimes of disorder parameters considered here. However, the reflection coefficient (r) distribution does not show much variation with the strength of the disorder parameter. In fact, they are quite similar. We will emphasize this point again when we later discuss the one-parameter scaling theory. The behavior of the probability distribution indicates that the distribution of the phase has little effect on the distribution of the reflection coefficient $P(r)$, and hence on the average transport properties.

C. Distribution of the phase (θ) with the strength of disorder parameter $2k\xi$

We now study the phase distribution $P(\theta)$ with the strength of the disorder parameter ($2k\xi$). We consider here mainly two cases of fixed lengths: (1) $l = 1$, and (2) the asymptotic limit when $l \to \infty$.

1. The phase (θ) distribution $P(\theta)$ for different disorder parameters ($2k\xi$) for fixed length $l = 1$.

Fig.6 shows the plot of $P(\theta)$ for the fixed length $l = 1$, with different values of the disorder parameters ($2k\xi$). Figures show that in the strong as well as in the weak disorder limit $P(\theta)$ is insensitive to the strength of the disorder parameter. Phase distribution is uniform for the weak disorder case. It has a perfectly symmetric peak centered at $\theta = \pi$ for the strong disorder case. In the case of intermediate disorder, the distributions are asymmetric with respect to the $\theta = \pi$ point. For the medium disorder strength parameter, $P(\theta)$ interpolates continuously between the strong and the weak disorder limiting phase distributions. It is clear from the $P(\theta)$ distributions, that the random phase approximation is valid for the condition $\xi > \lambda = 2\pi/k$, where λ is the wave length of the incoming wave and ξ is the localization length. (This has been checked systematically in our numerical calculation.)

2. The phase (θ) distribution $P(\theta)$ in the asymptotic limit of large lengths

Solution of the FP equation (Eq.12) gives the full probability distribution $P(r,\theta)$ in the (r,θ)-space. For larger lengths, one can make the approximation: $r \approx 1$ and $P(r) \approx \delta(r-1)$. In this asymptotic limit, marginal distribution for phase $P(\theta)$ becomes:

$$P(\theta) = \int_0^1 P(r,\theta)\delta(r-1)dr \equiv P(1,\theta). \tag{26}$$

Now from Eq.12 and Eq.26 we get:

$$\frac{\partial P(\theta)}{\partial l} = \frac{\partial}{\partial\theta}\left[(1+\cos\theta)\frac{\partial}{\partial\theta}(1+\cos\theta)\right]P(\theta) - 2k\xi\frac{\partial P(\theta)}{\partial\theta}. \tag{27}$$

Fig.7 shows the plot of $P(\theta)$ in the asymptotic limit for different disorder strength parameters ($2k\xi$), as the parameter varies from the weak to the strong disorder limits. Pictures show that in the interval from 0 to 2π a symmetric double peaked distribution around π in the strong disorder regime; a uniform phase distribution in the weak disorder regime; and an asymmetric phase distribution in the intermediate disorder regime. These distributions indicate that the essential features of $P(\theta)$ have not changed with the $r \approx 1$ approximation, relative to the $l = 1$ case which is shown in Fig.6.

D. Discussions on the phase distribution

We saw that the random phase approximation (RPA) is valid for $\xi > \lambda$. Physical meaning of the RPA is that the incoming wave has to undergo multiple reflections before escaping a localization length and in the process the wave randomizes its phase. For the weak disorder case the localization length is large. Hence the phase of the wave gets randomized. In the other extreme case of

the strong disorder, system tries to behave as a perfect reflector. Hence, phase of the reflected wave tries to peak at π (opposite phase with respect to the incoming wave). In the regime of intermediate disorder, $P(\theta)$ distribution is disorder specific and has some bias points to peak.

At this point let us discuss the results of other workers. Phase distribution for 1D systems have been studied earlier by Sulem [23], Stone *et. al.* [24], Jayannavar [25], Heinrichs [26] and Manna *et. al.* [27]. Sulem missed the random phase distribution in the limit of weak disorder, and his calculated phase distributions show peak near $\pm\pi$, and it does not look like a symmetric distribution. The study of Stone *et. al.* showed for the 1D Anderson model that (i) for the case of weak disorder the phase distribution is uniform, (ii) non-uniform for the strong disorder, (iii) and pinning of phase distribution near 2π for very strong disorder and large lengths. Their works also confirm that the distribution of the phase is insensitive to the disorder strength in the limit of weak as well as strong disorder limits. Manna *et. al.* show uniform phase distribution from 0 to 2π for the weak disorder, peaking of the phase distribution near π for the strong disorder. Jayannavar's calculation for the asymptotic ($l \to \infty$) phase distribution shows a uniform distribution of phase for weak disorder. However, the phase distribution for strong disorder shows several peaks that do not agree with our results, though we have numerically solved the same FP equation.

E. Marginal probability distribution $P(r)$ of the reflection coefficient r with respect to the disorder strength parameter $2k\xi$

Fig.8 shows the probability distribution of the reflection coefficient with the disorder parameter strength ($2k\xi$) for the sample length $l = 1$. It shows very little dependence on the strength of disorder parameter. But the distribution certainly has a small spread.

We can see that though the phase (θ) distributions $P(\theta)$ is quite different for different strengths of the disorder parameter ($2k\xi$), the reflection coefficient (r) distribution $P(r)$ does not change appreciably with the disorder strength for a fixed length of the sample.

F. Probability distributions for: g, ρ, $ln(g)$ and $ln(\rho)$

Once the marginal distribution $P(r)$ is known, the probability distribution of any quantity which is a function of r, can be easily calculated through the Jacobian of the probability transformation. Thus, from the Landauer four-probe resistance formula Eq.(14) one can easily calculate the probability distributions for the resistance(ρ), conductance(g), $ln\rho$ and lng , by multiplying with the proper Jacobian in to $P(r)$. We can write:

$$P(\rho) = P(r)(1-r)^2, \tag{28}$$

$$P(g) = P(r)r^2, \tag{29}$$

$$P(ln(\rho)) = P(r)r(1-r), \tag{30}$$

$$P(-ln(g)) = P(r)r(1-r). \tag{31}$$

(Here we have used the same symbol P for before and after the transformation.)
Fig.9 shows the plot of $P(\rho\ )$, and
Fig.10 shows the plot of $P(g)$, for the a fixed length $l = 10$, and for the three different regimes of disorder. These distributions show a log-normal form for all disorder strengths.

Fig.11 shows plots of $P(ln(\rho))(\equiv P(-ln(g)))$ vs $ln(\rho)$ ($\equiv -ln(g)$), for the fixed length $l = 10$. These distributions are Gaussian. (Probability distribution for the $ln(\rho)$ or $-ln(g)$ being a Gaussian implies that the ρ and g obey a log-normal distribution.)

Fig.12 shows the $ln(\rho)(\equiv -ln(g))$ distribution for the medium disorder with disorder parameter $2k\xi = 1$, for different lengths l. The distribution matches with the Gaussian form quite well even for the case of $l = 1$.

We can conclude that the probability distributions for ρ and g are log-normal even for a short sample length ($\sim$ localization length). They show little dependence on the strength of the disorder. The nature of the distribution is not affected by the phase fluctuations.

G. Mean and rms fluctuation of r, g and ρ

Figs.13(a),(b),(c), show the plots of the averages and the root-mean-squared fluctuations of r, ρ and g against the sample length l for different disorder strength parameters. From the figures one can see $< r >$ first increases rapidly and then saturates slowly to $r = 1$ for the large lengths. The average resistance ($< \rho >$) shows exponential increase and the average conductance ($< g >$) shows an exponential decrease with the sample length l. The fluctuations in the case of ρ and g are always more than the average as shown in the figure. However, in case of r the fluctuations are always less than the average and are finite. At this point we would like to mention that the average quantities like average resistance do not admit proper treatment by this type of numerical calculation for large length. This is because the Landauer four-probe resistance formula is $\rho = \frac{r}{1-r}$, that is, ρ is singular at r=1. The $P(r)$ never saturates (i.e., no steady state solution), but is localized near $r = 1$ for large lengths. For larger lengths, contribution to $< \rho >$ is dominated by the r-values very near to $r = 1$ ($\equiv \rho \to \infty$) singularity, and numerically it is very difficult to include this long-tail $\rho - distribution$ for the resistance. However, numerical calculation of the probability distribution of a quantity which is a non-singular function of r, does not pose much problem.

H. One-parameter scaling theory and the phase

Here we are re-examining the effect of disorder strength and phase distribution on the one-parameter scaling theory of localization. The scaling theory has been discussed in Ref. [28,29]. According to the ansatz of the scaling theory, the dimension-less conductance g (in units of $e^2/\pi\hbar$) obeys a universal scaling relation,

$$\frac{dln(g)}{dln(l)} = \beta(g). \tag{32}$$

The above scaling function $\beta(g)$ depends only on the dimensionality. The scaling relation suggest that the fractional change of the conductance per unit fractional change of the length depends only on g at that length; that is, once g is known at any length scale, one can derive the conductance for all lengths. The central idea of the scaling theory lies in the definition of the β-function which is assumed to have a smooth one-parameter scaling behavior with the system size. Now, it is well established that these assumptions are in general not correct because the scaling theory neglects fluctuations. For the case of pure elastic scattering, the resistance/conductance is not a self-averaging quantity. Fluctuations increase with system size for insulating regime, and in the metallic regime there is UCF, which has a finite non-zero value. These issues have been debated by several authors [30–33].
The points are the following:
(1) The precise meaning of the scaling function $\beta(g)$ is not very clear,
(2) g is a random quantity which depends on the microscopic details of the impurities inside the localization length, and only for length scales larger than the localization length g may have some universal properties. Only defining g, independent of the microscopic details, will not specify the average behavior of the sample.
The question then is how to define a scaling function which can accommodate fluctuation effects too. It is meaningful to define statistical properties like the probability distribution of the resistance/conductance in terms of which one can calculate the average properties of the system. One can then look for the number (one,or more) of parameters needed to specify the probability distribution [16,34].

There have been several studies [10,16,34,35] that show that for 1D in the weak disorder limit (i.e. within RPA) , $ln(\rho)$ has a normal distribution (we also have discussed the same in the earlier Section), in the limit of large lengths.

Now, the expression for the Gaussian distribution for logarithm of the resistance will be:

$$P(\ln\rho) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[\frac{-(ln\rho - < ln\rho >)^2}{2\sigma^2}\right] \tag{33}$$

which requires two parameters: the mean $< ln\rho >$ and the standard deviation σ.

It was found [36] that in the random phase approximation (RPA) the nth cumulant C_n for the large lengths l,

$$C_n(\ln\rho) \to la_n, \tag{34}$$

where a_n is some constant. It has also been seen by an exact solution for the weak disorder case [10] that,

$$C_1 = a_1 l \quad \text{and} \quad C_2 = a_2 l \tag{35}$$

Eq. (33) can now be rewritten as:

$$P(\ln\rho) = \frac{1}{\sqrt{4\pi a_2 l}} exp\left[\frac{-(ln\rho - a_1 l)^2}{2a_2 l}\right] \tag{36}$$

The above Gaussian distribution has clearly two parameters ($a1$ and $a2$) but can be reduced to a single parameter if the average and the rms fluctuations (variance) are universally related. Knowing the probability distributions $P(r)$ for different lengths and disorder strengths, we can check the validity of the one-parameter scaling theory for the probability distributions $P(ln\rho)$.

In Fig.14 we have plotted $var\, ln\rho$ vs $< ln\rho >$ for the cases of(1) weak, (2) medium, and (3) strong disorder. To obtain these plots we have calculated $< ln\rho >$ and $var\ (ln\rho)$, for a chosen value of the disorder parameter $2k\xi$, as function of the sample length. Different plots thus correspond to the different value of the disorder parameter strengths. A One-parameter scaling would have these plots coincide. The figure shows that the $var\ (ln\rho)$ vs $< ln\rho >$ plots follow approximately the same graph for the different disorder strengths and nearly coincide for large lengths. To this extent we have a deviation from the one-parameter scaling theory in 1D.

VIII. DISCUSSION AND CONCLUSIONS

We have solved here the 1D transport problem for the case of Gaussian white-noise disorder numerically. This is a nearly complete solution of transport properties of the 1D Gaussian white-noise random potential that goes beyond the conventional random phase approximation (RPA) valid only for weak disorder. We have evolved the full probability distribution in the reflection coefficient(r) and the associated phase (θ) space (i.e. (r,θ)-space) of the complex reflection amplitude $R = \sqrt{(r)}e^{i\theta}$ for a 1D disordered sample, with different lengths and with different disorder strengths. For our numerical solution, we have taken a fixed initial reflection coefficient $r_0 = .01$ for all realization of disorder as the Fokker-Planck equation for $P(r,\theta)$ is singular at $r = 0$. Gross statistical properties of the system are not expected to change with this weak extra scatterer. It may, however, affect the very sensitive details corresponding to the limit $r \to 0$, $l \to 0$. Our numerical work is a systematic study to observe the

contribution of the phase fluctuations to different averages.
On the basis of the results obtained, our conclusions are the following:

(A) $P(r,\theta)$ and $P(\theta)$ distributions.
1. The probability distribution $P(r,\theta)$ obeys two-parameter scaling to include the phase fluctuations and merge to one-parameter scaling in the limit of weak disorder. Full distribution of $P(r,\theta)$ can be described by the length l and disorder parameter $2k\xi$.
2. Phase distributions $P(\theta)$ also have the same two-parameter scaling dependence for arbitrary disorder strengths; and uniform phase distribution in the limit of weak disorder, i.e., one-parameter scaling.

(B) Phase θ distribution $P(\theta)$ in different regimes of disorder.
1. The random phase approximation (RPA) implying uniform phase distribution over 2π is valid for the condition $\xi/\lambda > 1$ (ξ is localization length and λ is incoming wavelength), that is, in the weak disorder limit. Physically, this means that the wave has to undergo multiple reflections before it moves through one localization length. $P(\theta)$ is independent of the disorder in the weak disorder limit ($2k\xi >> 1$).
2. In the strong disorder regime the distribution of the phase is perfectly symmetric in the interval from 0 and 2π, centering at $\theta = \pi$. The distribution is independent of the disorder in the strong disorder limit ($2k\xi << 1$).
3. In the intermediate disorder regime, phase distribution is asymmetric about $\theta = \pi$ point in the interval 0 and 2π . Also, the distributions are strongly disorder dependent.

(C) Reflection Coefficient (r) distributions $P(r)$:
The probability distribution for the reflection coefficient peaks at $r = 1$ for large lengths $l \equiv L/\xi >> 1$
Though probability distribution $P(\theta)$ for the phase varies qualitatively with the variation of the disorder strength, the probability distribution for the reflection coefficient $P(r)$ does not change qualitatively with the disorder strength for a fixed length of the sample.

(D) Distribution of ρ, g, $ln(\rho)$, $ln(g)$:
For large lengths of the sample (larger than the localization length) the probability distribution for ρ/ g obeys the log-normal distribution (i.e., $ln(\rho)/ln(g)$ obeys the normal distribution) for all disorder strengths. An asymmetric phase distribution does not affect the nature of the probability distribution of the resistance, or the conductance, which stays essentially log-normal.

(E) The mean and the rms fluctuations of ρ and g:
1.The average resistance increases exponentially with the length of the sample.
2.The average conductance decreases exponentially with the sample length.
3. The rms fluctuations are more than the average for both, the resistance and the conductance.

(F) The One-Parameter Scaling theory and the phase
The one-parameter scaling theory is approximately valid for the resistance and the conductance with $ln(\rho)$ *or* $ln(g)$ as the correct scaling variable for large sample lengths.

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FIG. 1. Initial probability distribution, i.e., when the sample length $l = 0$: (a) Probability distribution $P(r, \theta)$. (b) Marginal probability distribution $P(r)$. (c) Marginal probability distribution $P(\theta)$.

FIG. 2. Evolution of the probability $P(r, \theta)$ with the sample length l in the weak disorder regime for a fixed disorder strength parameter $2k\xi = 100$. Plots are for sample lengths: (a) $l = 1$, (b) $l = 2$, (c) $l = 5$, and (d) $l = 10$.

FIG. 3. Evolution of the probability $P(r, \theta)$ with the sample length l in the medium disorder regime for a fixed disorder strength parameter $2k\xi = 1$. Plots are for sample lengths: (a) $l = 1$, (b) $l = 2$, (c) $l = 5$, and (d) $l = 10$.

FIG. 4. Evolution of the probability $P(r, \theta)$ with the sample length l in the strong disorder regime for a fixed disorder strength parameter $2k\xi = .001$. Plots are for sample lengths: (a) $l = 1$, (b) $l = 2$, (c) $l = 5$, and (d) $l = 10$.

FIG. 5. Marginal probability distribution $P(r)$ and $P(\theta)$ separately against the sample length l. Three plots are for the fixed disorder strength parameters: (a) Weak disorder, $2k\xi = 100$, (b) Medium disorder, $2k\xi = 1$, and (c) Strong disorder, $2k\xi = .001$ (corresponding to Figs.2, 3 and 4).

FIG. 6. Phase θ distribution $P(\theta)$ against the disorder parameter $2k\xi$ for a fixed sample length $l = 1$.

FIG. 7. Phase θ distribution $P(\theta)$ against the disorder parameter $2k\xi$ for the asymptotic limit of large length $l \to \infty$.

FIG. 8. Reflection coefficient r distribution $P(r)$ against the disorder parameter $2k\xi$ for a fixed sample length $l = 1$.

FIG. 9. The resistance ρ distribution $P(\rho)$ against the disorder strength parameter $2k\xi$ for a fixed sample length $l = 10$.

FIG. 10. The conductance g distribution $P(g)$ against the disorder strength parameter $2k\xi$ for a fixed sample length $l = 10$.

FIG. 11. Logarithm of the resistance/conductance $ln\rho/lng$ distribution $P(ln\rho)/P(lng)$ against the disorder strength parameter $2k\xi$ for a fixed sample length $l = 10$.

FIG. 12. Logarithm of the resistance/conductance $ln\rho/lng$ distribution $P(ln\rho)/P(lng)$ against the sample length l in the medium disorder regime for a fixed disorder strength parameter $2k\xi = 1$.

FIG. 13. Plot of the average and the rms fluctuation versus length l for the different disorder strength parameters $2k\xi$ for (a) Reflection coefficient r, (b) Resistance ρ and (c) Conductance g.

FIG. 14. Plot of the $var\ \ ln\rho$ vs $< ln\rho >$ for different disorder strengths $2k\xi$.